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EQUIVARIANT MAPS BETWEEN REPRESENTATION SPHERES OF CYCLIC p-GROUPS

Ko OHASHI

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Abstract

This paper deals with necessary conditions for the existence of equivariant maps between the unit spheres of unitary representations of a cyclic p-group G. T. Bartsch gave a necessary condition for some unitary representations of G by using equivariant K-theory. We give two necessary conditions following Bartsch's approach. One is a generalization of Bartsch's result for any unitary representation of G which does not contain the trivial representation. The other is a stronger necessary condition for some special cases.

1. Introduction

The Borsuk-Ulam theorem asserts that if there exists a continuous map from S^m to S^n commuting with the antipodal map, then $n - m$ is greater than or equal to 0. One way to generalize this theorem is to consider equivariant maps between the unit spheres of representations of a given group. The unit sphere of a representation is called a representation sphere.

Bartsch [2] showed the following theorem for some unitary representation spheres of a cyclic p-group. (Note that Vick [10], Munkholm and Nakaoka [8] also obtained a similar result for equivariant maps from a lens space to a representation sphere.) Let p be a prime, N be a positive integer, and G be the cyclic group of order p^N with a generator g_0. For an integer α, let U_α be the 1-dimensional unitary representation $(\mathbb{C}, \rho_\alpha)$ of G which is defined by $\rho_\alpha : G \to \mathbb{C}^\times$ with $\rho_\alpha(g_0) = \zeta^\alpha$, where ζ is the complex number $\exp(2\pi\sqrt{-1}/p^N)$. We denote by $U_{\alpha,k}$ the direct sum of k-copies of U_α and $S(U_{\alpha,k})$ the unit sphere of $U_{\alpha,k}$.

Theorem 1.1 ([10] Corollary 3.3, [8] Theorem 4, [2] §§2 and 3)**.** *If there exists a G-equivariant map from $S(U_{1,m})$ to $S(U_{p^{N-1},n})$, then we have*

$$p^{N-1}(n-1) - (m-1) \geq 0.$$

Bartsch proved Theorem 1.1 by using equivariant K-theory and the K-theory Euler classes. In the study of equivariant maps between representation spheres, this method was originally used by Atiyah and Tall [1], and developed by Liulevicius [6], Bartsch [2], and Komiya [3, 4, 5].

In this paper, we give two necessary conditions for the existence of G-equivariant maps between unitary representation spheres of G. One is a generalization of Theorem 1.1 for any unitary representations of G which does not contain the trivial representation. The other

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is a stronger estimate for some special cases. In the proof we use equivariant K-theory, the K-theory Euler classes and elementary properties of cyclotomic polynomials following Bartsch [2] and Komiya [3, 5].

This paper is organized as follows. In §2, we prepare notations and state our main results. The rest of this paper is devoted to the proof of the results. In §3.1, we recall a theorem of Atiyah and Tall, and apply this theorem to unitary representation spheres of cyclic p-groups (Proposition 3.4). This yields a necessary condition for the existence of equivariant maps between them. Then we state Propositions 3.5 and 3.6, which give explicit consequences from the condition in Proposition 3.4. In the rest of §3, we prove the main results assuming Propositions 3.5 and 3.6. Section 4 is devoted to the proofs of Propositions 3.5 and 3.6. In the appendix, we collect properties of cyclotomic polynomials which are used throughout this paper.

2. Main results

In this paper we use the following notations. Let p be a prime and let G be the cyclic group of order p^N, $N \geq 1$. Let V and W be non-zero unitary representations of G with $V^G = W^G = 0$. For $0 \leq i \leq N$, let C_{p^i} denote the unique subgroup of G of order p^i. The unit spheres of V and W will be denoted by $S(V)$ and $S(W)$ respectively and the sets of all fixed points of V and W by the action of C_{p^i} will be denoted by $V^{C_{p^i}}$ and $W^{C_{p^i}}$ respectively. We use the symbol φ to denote Euler's phi function and use the symbol v_p to denote the p-adic valuation. Let $\Phi_d(x)$ denote the d-th cyclotomic polynomial and let $\phi^{(n)}_{p^a,i}$ denote the coefficients of the following expansion:

$$\Phi_{p^a}(x)^n = \sum_{i=0}^{n\varphi(p^a)} \phi^{(n)}_{p^a,i}(x-1)^i.$$

DEFINITION 2.1. (1) We define an integer N_V to be the largest integer n such that $V^{C_{p^n}}$ is not zero.

(2) For $0 \leq i \leq N$, let $d_i(V)$ be the complex dimension of $V^{C_{p^i}}$.

(3) We define an integer $d(V)$ by

$$d(V) := \sum_{i=0}^{N_V} \varphi(p^i)(d_i(V) - 1).$$

REMARK 2.2. The integer $d(V)$ satisfies $d(V) \geq \dim_{\mathbb{C}} V - 1$ and if G acts freely on $S(V)$, then $d(V) = \dim_{\mathbb{C}} V - 1$. Note that we assume V is not zero.

The following theorems are main results of this paper. Theorem 2.3 is a generalization of Theorem 1.1.

Theorem 2.3. *If there exists a G-equivariant map from $S(V)$ to $S(W)$, then we have*

$$d(W) - d(V) \geq 0.$$

Theorem 2.4. *Suppose $d_0(V) > d_0(W)$ and suppose that there exists an integer a such that*

(1) $1 \leq a \leq N_W$,

(2) $d_i(V) = d_i(W)$ *for* $i \neq 0, a$,
(3) $p^{N_W-(a-1)} \geq N - N_W$.

Let m and n be the integers defined by

$$m = d_0(V) - d_0(W), \quad n = d_a(W) - \max\{d_a(V), 1\}.$$

If there exists a G-equivariant map from $S(V)$ *to* $S(W)$*, then the integer n is positive and we have*

$$v_p(\phi^{(n)}_{p^a,m-1}) \geq N - N_W.$$

In the case $a = 1$ and $N - N_W = 2$, we obtain the following explicit estimate.

Corollary 2.5. *Suppose* $N - N_W = 2$ *and suppose that V and W satisfy*
(1) $d_0(V) > d_0(W)$,
(2) $\max\{d_1(V), 1\} \not\equiv d_1(W) \mod p$,
(3) $d_i(V) = d_i(W)$ *for* $i = 2, \ldots, N$.

If there exists a G-equivariant map from $S(V)$ *to* $S(W)$*, then we have*

$$d(W) - d(V) \geq \varphi(p).$$

REMARK 2.6. In the case $N = 2$, Stolz [9] and Meyer [7] gave stronger results than Theorem 2.3 by using stable cohomotopy theory. More precisely, let $v_{p,N}(m)$ (resp. $s_k(m)$) define to be the minimum number n such that there exists a G-equivariant map from $S(U_{1,m})$ to $S(U_{p^{N-1},n})$ (resp. S^{n-1}). Here G acts on S^{n-1} by the antipodal map. Stolz showed that $s(1) = 1$ and $s_2(m)$, $m \geq 2$ are given by

$$s_2(m) = \begin{cases} m+1 & \text{if } m \equiv 0, 2 \bmod 8, \\ m+2 & \text{if } m \equiv 1, 3, 4, 5, 7 \bmod 8, \\ m+3 & \text{if } m \equiv 6 \bmod 8. \end{cases}$$

For an odd prime p, Meyer showed that $v_{p,2}(1) = 1$ and $v_{p,2}(m)$, $m \geq 2$ satisfies

$$\left\langle \frac{m-2}{p} \right\rangle + 1 \leq v_{p,2}(m) \leq \left\langle \frac{m-2}{p} \right\rangle + 2 \qquad \text{if } m \not\equiv 2 \bmod p,$$

$$v_{p,2}(m) = \frac{m-2}{p} + 2 \qquad \text{if } m \equiv 2 \bmod p,$$

where the symbol $\langle x \rangle$ denotes the smallest integer bigger than or equal to x.

3. Proofs of main results

We prepare three propositions in §3.1 and by using the propositions, we prove Theorems 2.3, 2.4 in §§3.2, 3.3 and Corollary 2.5 in §3.4.

3.1. Reduction to algebraic problems. We recall the definition of the K-theory Euler class and a theorem of Atiyah and Tall [1]. For a complex representation U of a finite group H, the K-theory Euler class $e(U)$ of U is defined by the formula

$$e(U) = \sum_{i=0}^{\dim U} (-1)^i [\Lambda^i U] \in R(H),$$

where $R(H)$ denotes the complex representation ring of H and $[\Lambda^i U]$ is the isomorphism class of the i-th exterior power $\Lambda^i U$ of U. The next theorem is due to Atiyah and Tall.

Theorem 3.1 ([1], Part IV, §1). *Let V and W be unitary representations of H. If there exists an H-equivariant map from $S(V)$ to $S(W)$, then $e(V)$ divides $e(W)$ in $R(H)$.*

We will write down concretely the divisibility condition of the K-theory Euler classes of Theorem 3.1 for the cyclic group G of order p^N. In order to do this, we will use a ring isomorphism $f : R(G) \to \mathbb{Z}[x]/(x^{p^N} - 1)$ defined by

$$f([U_\alpha]) = [x^\alpha], \tag{3.1}$$

where U_α is the 1-dimensional unitary representation of G defined by the correspondence $g_0 \mapsto \zeta^\alpha$, where g_0 is a generator of G and ζ is the complex number $\exp(2\pi\sqrt{-1}/p^N)$.

Lemma 3.2. *For any unitary representation U of G, there exists a unitary representation U' of G with the following properties:*

(1) *There exist G-equivariant maps from $S(U)$ to $S(U')$ and from $S(U')$ to $S(U)$.*
(2) *The K-theory Euler class $e(U')$ satisfies*

$$f(e(U')) = (-1)^{\dim U} \left[\prod_{i=0}^{N_U} \Phi_{p^i}(x)^{d_i(U)} \right].$$

Proof. Let a_i, $0 \le i \le N_U$ be the non-negative integers defined by

$$a_i := d_i(U) - d_{i+1}(U).$$

Note that the sequence $\{d_i(U)\}_{i=0}^{N_U}$ satisfies

$$d_0(U) \ge d_1(U) \ge \cdots \ge d_{N_U}(U) > d_{N_U+1}(U) = 0.$$

Then we define U' to be the unitary representations of G of the forms

$$U' = a_0 U_1 \oplus a_1 U_p \oplus \cdots \oplus a_{N_U} U_{p^{N_U}}.$$

First we show that the unitary representation U' satisfies the condition (1). Since $\{U_\alpha \mid 1 \le \alpha \le p^N\}$ gives a complete set of irreducible representations of G, we can take irreducible decomposition of U as follows:

$$U = U_{\alpha_1} \oplus \cdots \oplus U_{\alpha_{\dim U}}, \quad 1 \le \alpha_k \le p^N.$$

Note that U' can be written as

$$U' = U_{(\alpha_1)_p} \oplus \cdots \oplus U_{(\alpha_{\dim U})_p},$$

where $(\alpha_k)_p$ denotes the largest power of p that divides α_k. Since the correspondence $z \mapsto z^a$ defines a G-equivariant map $S(U_\gamma) \to S(U_\delta)$ for any integers γ, δ and a with $\delta \equiv a\gamma \bmod p^N$, we have G-equivariant maps

$$\varphi_k : S(U_{(\alpha_k)_p}) \to S(U_{\alpha_k}), \quad \psi_k : S(U_{\alpha_k}) \to S(U_{(\alpha_k)_p})$$

for $1 \leq k \leq \dim U$. The join of the equivariant maps φ_k, $1 \leq k \leq \dim U$ gives a G-equivariant map

$$\varphi : S(U') \cong S(U_{(\alpha_1)_p}) * \cdots * S(U_{(\alpha_{\dim U})_p}) \to S(U_{\alpha_1}) * \cdots * S(U_{\alpha_{\dim U}}) \cong S(U),$$

where $*$ denotes the topological join. A similar construction for $(\psi_k)_{k=1}^{\dim U}$ gives a G-equivariant map $\psi : S(U) \to S(U')$.

Next we show that the unitary representation U' satisfies the condition (2). From the multiplicativity of the K-theory Euler class, it is easy to see that

$$f(e(U')) = (-1)^{\dim U} \left[\prod_{k=0}^{N_U} (x^{p^k} - 1)^{a_k} \right].$$

Hence it is sufficient to show

$$\prod_{k=0}^{N_U} (x^{p^k} - 1)^{a_k} = \prod_{i=0}^{N_U} \Phi_{p^i}(x)^{d_i(U)}.$$

This equation follows from

$$\prod_{k=0}^{N_U} (x^{p^k} - 1)^{a_k} = \prod_{k=0}^{N_U} \prod_{j=0}^{k} \Phi_{p^j}(x)^{a_k} = \prod_{i=0}^{N_U} \Phi_{p^i}(x)^{a_i + \cdots + a_{N_U}} = \prod_{i=0}^{N_U} \Phi_{p^i}(x)^{d_i(U)}.$$

□

We also state the next lemma which we will use frequently in our argument.

Lemma 3.3. *Let R be an integral domain and a, x, y and z be elements of R.*

(1) *Suppose that a is not zero. Then $ax \in (ay, az)$ if and only if $x \in (y, z)$.*
(2) *Suppose that a is a prime element of R and $y \notin (a)$. Then $ax \in (y, az)$ if and only if $x \in (y, z)$.*

We omit the proof of this lemma since it is straightforward. From Theorem 3.1 and Lemma 3.2, we obtain the following proposition.

Proposition 3.4. *Let V and W be unitary representations of G with $V^G = W^G = 0$. If there exists a G-equivariant map from $S(V)$ to $S(W)$, then we have*

$$\prod_{j=0}^{N_W} \Phi_{p^j}(x)^{d'_j(W)} \in \left(\prod_{i=0}^{N_V} \Phi_{p^i}(x)^{d'_i(V)}, \Phi_{p^{N_W+1}}(x) \cdots \Phi_{p^N}(x) \right), \tag{3.2}$$

where $d'_i(V)$ and $d'_j(W)$ are defined by $d'_i(V) := d_i(V) - 1$, $d'_j(W) := d_j(W) - 1$.

Proof. From Lemma 3.2, there exist unitary representations V' and W' of G with the following properties:

(1) There exist G-equivariant maps from $S(V')$ to $S(V)$ and from $S(W)$ to $S(W')$.
(2) By the ring isomorphism f of (3.1), the K-theory Euler classes $e(V')$ and $e(W')$ correspond to

$$(-1)^{\dim V} \left[\prod_{i=0}^{N_V} \Phi_{p^i}(x)^{d_i(V)} \right], \quad (-1)^{\dim W} \left[\prod_{j=0}^{N_W} \Phi_{p^j}(x)^{d_j(W)} \right],$$

respectively.

Since we assume the existence of a G-equivariant map from $S(V)$ to $S(W)$, we obtain a G-equivariant map from $S(V')$ to $S(W')$. Theorem 3.1 implies that $e(V')$ divides $e(W')$ in $R(G)$. From the condition (2), we obtain

$$\prod_{j=0}^{N_W} \Phi_{p^j}(x)^{d_j(W)} \in \left(\prod_{i=0}^{N_V} \Phi_{p^i}(x)^{d_i(V)}, x^{p^N} - 1 \right). \tag{3.3}$$

By the existence of a G-equivariant map from $S(V)$ to $S(W)$ and $W^G = 0$, we have the inequalities $N_V \le N_W < N$. Then the required relation (3.2) follows immediately from repeated application of Lemma 3.3 to the relation (3.3). □

The next algebraic propositions give explicit consequences from the relation (3.2) in Proposition 3.4. The proofs are given in §4.

Proposition 3.5. *Let k, k' and ℓ be non-negative integers with $\max\{k, k'\} < \ell$ and let m_i and n_j be non-negative integers for $0 \le i \le k,\ 0 \le j \le k'$. Then*

$$\prod_{j=0}^{k'} \Phi_{p^j}(x)^{n_j} \in \left(\prod_{i=0}^{k} \Phi_{p^i}(x)^{m_i}, \Phi_{p^\ell}(x) \right) \tag{3.4}$$

if and only if

$$\sum_{i=0}^{k} \varphi(p^i) m_i \le \sum_{j=0}^{k'} \varphi(p^j) n_j.$$

Proposition 3.6. *Let N_1, N_2 and a be positive integers satisfying $a \le N_1 < N_2$ and $p^{N_1-(a-1)} \ge N_2 - N_1$. If non-negative integers m and n satisfy*

$$\Phi_{p^a}(x)^n \in (\Phi_1(x)^m, \Phi_{p^{N_1+1}}(x) \cdots \Phi_{p^{N_2}}(x)), \tag{3.5}$$

then

$$v_p\left(\phi^{(n)}_{p^a, m-1}\right) \ge N_2 - N_1.$$

Here we set $\phi^{(n)}_{p^a,-1} := 0$.

REMARK 3.7. In fact, the converse of Proposition 3.6 is also true. However we omit the proof of the converse since it is not needed for our purpose.

3.2. Proof of Theorem 2.3. From Proposition 3.4, we have

$$\prod_{j=0}^{N_W} \Phi_{p^j}(x)^{d'_j(W)} \in \left(\prod_{i=0}^{N_V} \Phi_{p^i}(x)^{d'_i(V)}, \Phi_{p^N}(x) \right).$$

Applying Proposition 3.5, we obtain the required inequality

$$d(V) = \sum_{i=0}^{N_V} \varphi(p^i) d'_i(V) \le \sum_{j=0}^{N_W} \varphi(p^j) d'_j(W) = d(W).$$

This completes the proof. □

3.3. Proof of Theorem 2.4. It follows from Proposition 3.4 that

$$\prod_{j=0}^{N_W} \Phi_{p^j}(x)^{d'_j(W)} \in \left(\prod_{i=0}^{N_V} \Phi_{p^i}(x)^{d'_i(V)}, \Phi_{p^{N_W+1}}(x) \cdots \Phi_{p^N}(x) \right)$$

In view of the assumptions $d_0(V) > d_0(W)$ and (2) of Theorem 2.4, it follows from Lemma 3.3 and $N_V \leq N_W$ that

$$\Phi_{p^a}(x)^{d'_a(W)} \in \left(\Phi_1(x)^{d_0(V)-d_0(W)} \Phi_{p^a}(x)^{\bar{d}_a(V)},\ \Phi_{p^{N_W+1}}(x) \cdots \Phi_{p^N}(x) \right), \tag{3.6}$$

where $\bar{d}_a(V)$ is defined to be the integer $\max\{d_a(V) - 1, 0\}$. We show that $d'_a(W)$ is greater than $\bar{d}_a(V)$. For otherwise, the relation (3.6) implies that

$$1 \in \left(\Phi_1(x)^{d_0(V)-d_0(W)} \Phi_{p^a}(x)^{\bar{d}_a(V)-d'_a(W)},\ \Phi_{p^N}(x) \right),$$

and hence $d_0(V) = d_0(W)$, contradicting the assumption $d_0(V) > d_0(W)$. Combining Lemma 3.3 and the inequality $\bar{d}_a(V) < d'_a(W)$, the relation (3.6) yields

$$\Phi_{p^a}(x)^{d'_a(W)-\bar{d}_a(V)} \in \left(\Phi_1(x)^{d_0(V)-d_0(W)}, \Phi_{p^{N_W+1}}(x) \cdots \Phi_{p^N}(x) \right). \tag{3.7}$$

Note that

$$d'_a(W) - \bar{d}_a(V) = d_a(W) - \max\{d_a(V), 1\}.$$

Applying Proposition 3.6 to (3.7), we obtain the required inequality. □

3.4. Proof of Corollary 2.5. For simplicity of notation, let $\tilde{d}_1(V)$ stand for the integer $\max\{d_1(V), 1\}$. From Theorem 2.4 (the case $a = 1$) and the assumption $N - N_W = 2$, it follows

$$v_p\left(\phi^{(n)}_{p,m-1}\right) \geq 2,$$

where m and n is given by $m = d_0(V) - d_0(W)$ and $n = d_1(W) - \tilde{d}_1(V)$. Lemma A.3(2) implies

$$\varphi(p)(d_1(W) - \tilde{d}_1(V) - \delta_{d_1(W)-\tilde{d}_1(V)}) \geq d_0(V) - d_0(W), \tag{3.8}$$

where $\delta_{d_1(W)-\tilde{d}_1(V)}$ is given by

$$\delta_{d_1(W)-\tilde{d}_1(V)} = \begin{cases} 0 & \tilde{d}_1(V) \equiv d_1(W) \mod p \\ 1 & \tilde{d}_1(V) \not\equiv d_1(W) \mod p. \end{cases}$$

The assumption (2) in Theorem 2.4 implies

$$\delta_{d_1(W)-\tilde{d}_1(V)} = 1. \tag{3.9}$$

By the definitions of $d(V)$ and $d(W)$, we have

$$d(W) - d(V) = d_0(W) - d_0(V) + \varphi(p)(d_1(W) - \tilde{d}_1(V)). \tag{3.10}$$

Combining (3.9) and (3.10) with (3.8), we obtain

$$d(W) - d(V) \geq \varphi(p),$$

as required. □

We have completed the proof of Theorems 2.3, 2.4 and Corollary 2.5 assuming Propositions 3.5 and 3.6.

4. Proofs of Propositions

In this section, we prove Propositions 3.5 and 3.6, which are used in §3.

4.1. Proof of Proposition 3.5. Let ζ_{p^ℓ} denote the complex number $\exp(2\pi\sqrt{-1}/p^\ell)$. Using a ring isomorphism $\mathbb{Z}[x]/(\Phi_{p^\ell}(x)) \to \mathbb{Z}[\zeta_{p^\ell}]$ defined by $[x] \mapsto \zeta_{p^\ell}$, we can reformulate the relation (3.4) in Proposition 3.5 as

$$\prod_{j=0}^{k'} \Phi_{p^j}(\zeta_{p^\ell})^{n_j} \in \left(\prod_{i=0}^{k} \Phi_{p^i}(\zeta_{p^\ell})^{m_i} \right). \tag{4.1}$$

It follows from Lemma A.1 that the relation (4.1) is equivalent to

$$\Phi_1(\zeta_{p^\ell})^{\sum_{j=0}^{k'} \varphi(p^j) n_j} \in (\Phi_1(\zeta_{p^\ell}))^{\sum_{i=0}^{k} \varphi(p^i) m_i}. \tag{4.2}$$

Since $\Phi_1(\zeta_{p^\ell})$ is not a unit of $\mathbb{Z}[\zeta_{p^\ell}]$, the relation (4.2) is equivalent to

$$\sum_{i=0}^{k} \varphi(p^i) m_i \le \sum_{j=0}^{k'} \varphi(p^j) n_j.$$

This completes the proof. □

4.2. Proof of Proposition 3.6. Let S be the set of all pairs of non-negative integers (m, n) such that

$$\Phi_{p^a}(x)^n \in (\Phi_1(x)^m, \Phi_{p^{N_1+1}}(x) \cdots \Phi_{p^{N_2}}(x)).$$

For a positive integer $\alpha \ge 1$, let $I_p^{(\alpha)}$ denote the integer $\varphi(p^{N_1+1}) + \cdots + \varphi(p^{N_1+\alpha})$ and we set $I_p^{(0)} := 1$. We define a sequence of integers $\{a_j\}_{j=0}^{I_p^{(N_2-N_1)}}$ and sequences of rational numbers $\{b_k\}_{k=0}^{\infty}$, $\{c_\ell(n)\}_{\ell=0}^{\infty}$ as the coefficients of the following expansions:

$$\begin{aligned}
\prod_{i=N_1+1}^{N_2} \Phi_{p^i}(x) &= \sum_{j=0}^{I_p^{(N_2-N_1)}} a_j (x-1)^j \in \mathbb{Z}[x], \\
\frac{1}{\prod_{i=N_1+1}^{N_2} \Phi_{p^i}(x)} &= \sum_{k=0}^{\infty} b_k (x-1)^k \in \mathbb{Q}[[x]], \\
\frac{\Phi_{p^a}(x)^n}{\prod_{i=N_1+1}^{N_2} \Phi_{p^i}(x)} &= \sum_{\ell=0}^{\infty} c_\ell(n)(x-1)^\ell \in \mathbb{Q}[[x]].
\end{aligned}$$

Here $\mathbb{Q}[[x]]$ is the ring of formal power series with coefficients in $\mathbb{Q}$. The next lemma follows immediately from the definitions of S and $\{c_\ell(n)\}_{\ell=1}^{\infty}$.

Lemma 4.1. *$(m, n) \in S$ if and only if $c_0(n), \ldots, c_{m-1}(n)$ are integers.*

From Lemma A.4(1), it is easy to see the following lemma.

Lemma 4.2. *If $I_p^{(\alpha)} \le j < I_p^{(\alpha+1)}$, then $v_p(a_j) \ge N_2 - (N_1 + \alpha)$.*

For an integer k and a positive integer w, let $q(k,w)$ be the unique integer satisfying the inequality

$$q(k,w)\varphi(p^w) \leq k < \{q(k,w)+1\}\varphi(p^w).$$

Lemma 4.3. *The number $p^{N_2-N_1+q(k,N_1+1)}b_k$ is an integer for any $k \geq 0$.*

Proof. We will prove the statement by induction on $k \geq 0$. When $k = 0$, this follows from

$$p^{N_2-N_1}b_0 = a_0b_0 = 1 \in \mathbb{Z}.$$

Let $k \geq 1$ and suppose that the assertion is true up to $k-1$. By the definitions of $\{a_j\}_{j=0}^{I_p(N_2-N_1)}$ and $\{b_k\}_{k=0}^{\infty}$, we have

$$p^{N_2-N_1}b_k = a_0b_k = -\sum_{j=1}^{k} a_jb_{k-j},$$

where we set $a_j := 0$ for $j > I_p^{(N_2-N_1)}$. Hence it is sufficient to show that $p^{q(k,N_1+1)}a_jb_{k-j}$ is an integer for $1 = I_p^{(0)} \leq j \leq I_p^{(N_2-N_1)}$.

Case 1. Suppose $I_p^{(0)} \leq j < I_p^{(1)}$. Lemma 4.2 implies

$$v_p(a_j) \geq N_2 - N_1.$$

On the other hand, the induction hypothesis implies that

$$p^{N_2-N_1+q(k,N_1+1)}b_{k-j} \in \mathbb{Z}.$$

Hence we have

$$p^{q(k,N_1+1)}a_jb_{k-j} \in \mathbb{Z}$$

for $I_p^{(0)} \leq j < I_p^{(1)}$.

Case 2. Suppose $I_p^{(\alpha)} \leq j < I_p^{(\alpha+1)}$ for some α with $1 \leq \alpha \leq N_2 - N_1$. Lemma 4.2 implies

$$v_p(a_j) \geq N_2 - (N_1 + \alpha).$$

On the other hand, since

$$\begin{aligned} k - j &< \{q(k,N_1+1)+1\}\varphi(p^{N_1+1}) - I_p^{(\alpha)} \\ &= \{q(k,N_1+1)+1-(1+p+\cdots+p^{\alpha-1})\}\varphi(p^{N_1+1}), \end{aligned}$$

we have

$$q(k-j,N_1+1) \leq q(k,N_1+1) - (1+p+\cdots+p^{\alpha-1}) \leq q(k,N_1+1) - \alpha,$$

and hence the induction hypothesis implies that

$$p^{q(k,N_1+1)+N_2-(N_1+\alpha)}b_{k-j} \in \mathbb{Z}.$$

Therefore we obtain

$$p^{q(k,N_1+1)}a_jb_{k-j} \in \mathbb{Z},$$

for $I_p^{(\alpha)} \leq j < I_p^{(\alpha+1)}$. □

Proof of Proposition 3.6 . We will prove the statement by induction on $m \geq 0$. When $m = 0$, the assertion is trivial. Let $m \geq 1$ and suppose that the assertion is true up to $m - 1$. We assume that $(m, n) \in S$. From the definitions of $\{b_k\}_{k=0}^{\infty}$ and $\{c_\ell(n)\}_{\ell=0}^{\infty}$, it follows

$$p^{-(N_2-N_1)}\phi^{(n)}_{p^a,m-1} = c_{m-1}(n) - \sum_{k=1}^{m-1} b_k \phi^{(n)}_{p^a,m-1-k}. \tag{4.3}$$

On the other hand, it follows from Lemma 4.1 that $c_{m-1}(n)$ is an integer. If we prove the inequalities

$$v_p\left(\phi^{(n)}_{p^a,m-1-k}\right) \geq N_2 - N_1 + q(k, N_1 + 1), \quad k \geq 1, \tag{4.4}$$

then combining Lemma 4.3 with the equation (4.3), we have

$$p^{-(N_2-N_1)}\phi^{(n)}_{p^a,m-1} \in \mathbb{Z}.$$

Hence it is sufficient to show (4.4).

Case 1. Suppose $k \geq I_p^{(1)}$. In this case, we have $q(k, N_1 + 1) \geq 1$. By Lemma A.4(2), we have

$$v_p\left(\phi^{(n)}_{p^a,m-1-k}\right) \geq n - q(m - 1 - k, a).$$

From the inequality $k \geq q(k, N_1 + 1)\varphi(p^{N_1+1})$, it follows

$$\begin{aligned} v_p\left(\phi^{(n)}_{p^a,m-1-k}\right) &\geq n - q\left(m - 1 - q(k, N_1 + 1)\varphi(p^{N_1+1}), a\right) \\ &= n - q(m - 1, a) + q(k, N_1 + 1)p^{N_1-(a-1)}. \end{aligned}$$

By the assumption $(m, n) \in S$ and Proposition 3.5, we have

$$n \geq q(m - 1, a) + 1.$$

Hence

$$v_p\left(\phi^{(n)}_{p^a,m-1-k}\right) \geq 1 + q(k, N_1 + 1)p^{N_1-(a-1)}.$$

From the assumption $p^{N_1-(a-1)} \geq N_2 - N_1$ and $q(k, N_1 + 1) \geq 1$, we obtain the required inequality

$$v_p\left(\phi^{(n)}_{p^a,m-1-k}\right) \geq q(k, N_1 + 1) + N_2 - N_1.$$

Case 2. Suppose $I_p^{(0)} \leq k < I_p^{(1)}$. In this case, the integer $q(k, N_1 + 1)$ is zero. Note that $(m, n) \in S$ implies $(m - k - 1, n) \in S$. Then the induction hypothesis implies that

$$v_p\left(\phi^{(n)}_{p^a,m-1-k}\right) \geq N_2 - N_1 = N_2 - N_1 + q(k, N_1 + 1).$$

□

Appendix A. Cyclotomic polynomials

This appendix will collect properties of cyclotomic polynomials which are used throughout this paper.

The following lemma is well-known.

Lemma A.1. *If k is less than ℓ, then $\Phi_{p^k}(\zeta_{p^\ell})$ and $\Phi_1(\zeta_{p^\ell})^{\varphi(p^k)}$ are associates in $\mathbb{Z}[\zeta_{p^\ell}]$, that is $\Phi_{p^k}(\zeta_{p^\ell}) = u\,\Phi_1(\zeta_{p^\ell})^{\varphi(p^k)}$ for some unit u of $\mathbb{Z}[\zeta_{p^\ell}]$.*

Next we give three lemmas on the p-adic valuation of $\phi^{(n)}_{p^a,i}$. Here $\phi^{(n)}_{p^a,i}$, $0 \le i \le n\varphi(p^a)$ are the coefficients of the following expansion:

$$\Phi_{p^a}(x)^n = \sum_{i=0}^{n\varphi(p^a)} \phi^{(n)}_{p^a,i}(x-1)^i.$$

Lemma A.2. *The integer $\phi^{(1)}_{p,i}$ is given by*

$$\phi^{(1)}_{p,i} = \binom{p}{i+1}.$$

In particular, it satisfies

$$v_p\left(\phi^{(1)}_{p,i}\right) = \begin{cases} 1 & 0 \le i < \varphi(p) \\ 0 & i = \varphi(p). \end{cases}$$

We omit the proof of this lemma since it is straightforward. Recall that we denote by $q(k,w)$ the unique integer satisfying the inequality

$$\varphi(p^w)q(k,w) \le k < \{q(k,w)+1\}\varphi(p^w).$$

Lemma A.3. (1) $v_p\left(\phi^{(n)}_{p,i}\right) \ge n - q(i,1)$.

(2) *Suppose $n \ge q(i,1)+1$. Then $v_p\left(\phi^{(n)}_{p,i}\right) \ge 2$ if and only if $\varphi(p)(n-\delta_n) > i$, where δ_n is given by*

$$\delta_n = \begin{cases} 1 & n \not\equiv 0 \mod p \\ 0 & n \equiv 0 \mod p. \end{cases}$$

Proof. We first show Lemma A.3(1). The product rule implies

$$\phi^{(n)}_{p,i} = \sum_{i_1,\dots,i_n} \phi^{(1)}_{p,i_1} \cdots \phi^{(1)}_{p,i_n}, \tag{A.1}$$

where the sum is taken over all the integers $i_1, \dots, i_n$ such that

$$i_1 + \cdots + i_n = i, \quad 0 \le i_j \le \varphi(p).$$

Then Lemma A.3(1) follows from (A.1) and the inequality

$$v_p\left(\phi^{(1)}_{p,i_1} \cdots \phi^{(1)}_{p,i_n}\right) = n - \#\{j \in \{1,\dots,n\} |\, i_j = \varphi(p)\} \ge n - q(i,1).$$

Note that the first equality is given by Lemma A.2.

Next we prove Lemma A.3(2) in two steps.

Step 1. The first step is to show the following equivalence:

$$v_p\left(\phi^{(n)}_{p,i}\right) \ge 2 \iff n \ge q(i,1)+1+\delta_{q(i,1)+1}.$$

From Lemma A.3(1) and the assumption $n \ge q(i,1)+1$ of Lemma A.3(2), it is sufficient to show

$$v_p\left(\phi^{(q(i,1)+1)}_{p,i}\right) \ge 2 \iff q(i,1)+1 \equiv 0 \mod p. \tag{A.2}$$

Let r be the integer $i - \varphi(p)q(i,1)$. From (A.1) (the case $n = q(i,1)+1$) and Lemma A.2, it follows

$$\phi_{p,i}^{(q(i,1)+1)} \equiv \{q(i,1)+1\}\phi_{p,r}^{(1)} \mod p^2.$$

Since $0 \le r < \varphi(p)$, Lemma A.2 implies $v_p(\phi_{p,r}^{(1)}) = 1$ and hence we have the equivalence (A.2).

Step 2. The second step is to show the following equivalence:

$$n \ge q(i,1)+1+\delta_{q(i,1)+1} \quad \Longleftrightarrow \quad \varphi(p)(n-\delta_n) > i.$$

Since the inequality $\varphi(p)(n-\delta_n) > i$ is equivalent to the inequality $n - \delta_n \ge q(i,1)+1$, it is sufficient to show

$$n \ge q(i,1)+1+\delta_{q(i,1)+1} \quad \Longleftrightarrow \quad n-\delta_n \ge q(i,1)+1.$$

This follows from the following general equivalence: $a \ge b+\delta_b$ if and only if $a - \delta_a \ge b$ for any integers a and b. This follows immediately from the definition of δ. □

Lemma A.4. (1) *If* $0 \le j < \varphi(p^a)$, *then* $v_p\left(\phi_{p^a,j}^{(1)}\right) \ge 1$.
(2) $v_p\left(\phi_{p^a,i}^{(n)}\right) \ge n - q(i,a)$.

Proof. First we prove Lemma A.4(1) by induction on $a \ge 1$. For simplicity of notation, we write $\phi_{p^a,j}$ instead of $\phi_{p^a,j}^{(1)}$. When $a = 1$, the statement follows immediately from Lemma A.2. Let $k \ge 2$ and suppose that the assertions are true for $a = k-1$. Let j be an integer such that $0 \le j < \varphi(p^k)$. The polynomial $\Phi_{p^k}(x)$ can be described as

$$\Phi_{p^k}(x) = \Phi_{p^{k-1}}(x^p) = \sum_{n=0}^{\varphi(p^{k-1})}\sum_{i=0}^{n\varphi(p)} \phi_{p^{k-1},n}\,\phi_{p,i}^{(n)}\Phi_1(x)^{n+i}.$$

This yields the formula

$$\phi_{p^k,j} = \sum_{n,i}\phi_{p^{k-1},n}\phi_{p,i}^{(n)},$$

where the sum is taken over all integers n and i such that

$$\text{(A.3)} \qquad 0 \le n \le \varphi(p^{k-1}),\ 0 \le i \le n\varphi(p),\ n+i = j.$$

In particular we obtain

$$\text{(A.4)} \qquad v_p(\phi_{p^k,j}) \ge \min\{v_p(\phi_{p^{k-1},n}) + v_p(\phi_{p,i}^{(n)}) \mid n \text{ and } i \text{ satisfy (A.3)}\}.$$

Hence it is sufficient to show that

$$v_p(\phi_{p^{k-1},n}) + v_p(\phi_{p,i}^{(n)}) \ge 1$$

for any integers n and i satisfying (A.3). The condition (A.3) implies

$$1 \le n < \varphi(p^{k-1}), \quad \text{or} \quad (\,n = \varphi(p^{k-1}) \text{ and } 0 \le i < n\varphi(p)\,).$$

When $1 \le n < \varphi(p^{k-1})$, the induction hypothesis for $a = k-1$ implies $v_p(\phi_{p^{k-1},n}) \ge 1$. When $n = \varphi(p^{k-1})$ and $0 \le i < n\varphi(p)$, Lemma A.3(1) implies that $v_p(\phi_{p,i}^{(n)}) \ge 1$. These inequalities implies that $v_p(\phi_{p^k,j}) \geq 1$. This completes the proof of Lemma A.4(1).

The proof of Lemma A.4(2) is similar to that of Lemma A.3(1), using Lemma A.4(1) instead of Lemma A.2. □

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